

The University of Chicago

SUFFICIENT CONDITIONS FOR A
MINIMUM IN THE PROBLEM OF
LAGRANGE WITH ISOPERI-
METRIC CONDITIONS

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1935

By
IRWIN EARL PERLIN

Private Edition, Distributed by
THE UNIVERSITY OF CHICAGO LIBRARIES
CHICAGO, ILLINOIS

Reprinted from
CONTRIBUTIONS TO THE CALCULUS OF VARIATIONS, 1933-37
The University of Chicago Press, 1937

The University of Chicago

SUFFICIENT CONDITIONS FOR A
MINIMUM IN THE PROBLEM OF
LAGRANGE WITH ISOPERI-
METRIC CONDITIONS

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1935

By
IRWIN EARL PERLIN

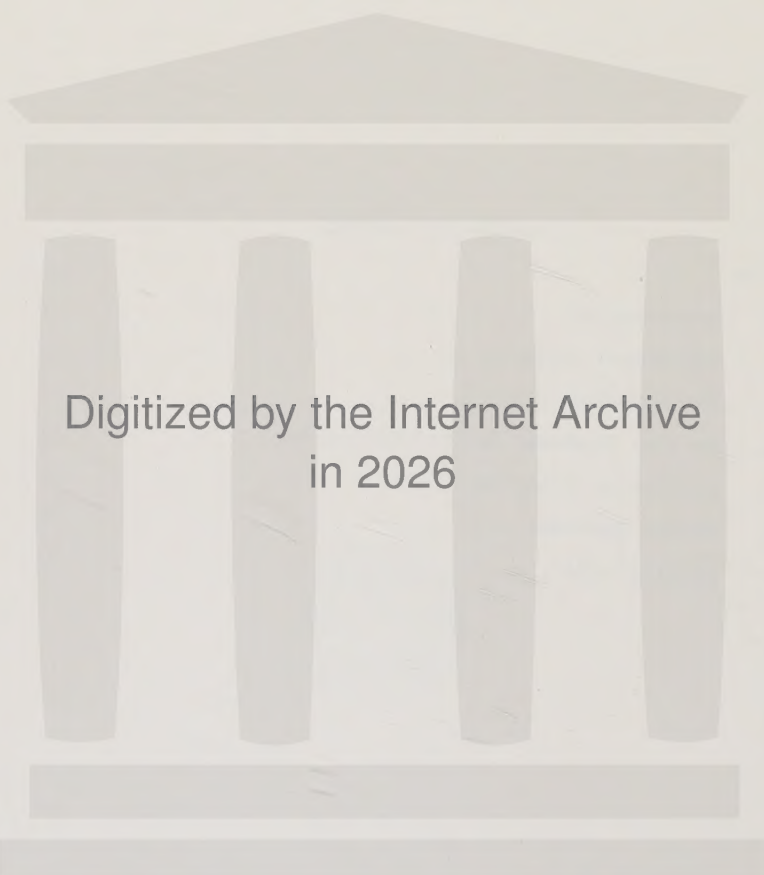
Private Edition, Distributed by
THE UNIVERSITY OF CHICAGO LIBRARIES
CHICAGO, ILLINOIS

Reprinted from
CONTRIBUTIONS TO THE CALCULUS OF VARIATIONS, 1933-37
The University of Chicago Press, 1937



TABLE OF CONTENTS

Section	Page
1. Introduction	1
2. Analytical Formulation	3
3. The Lindeberg Theorem	8
4. General Theorems on Relative Minima	17
5. Sufficient Conditions	23
6. Osgood Theorems	28
Bibliography	31



Digitized by the Internet Archive
in 2026

https://archive.org/details/IA41548407_0095

SUFFICIENT CONDITIONS IN THE PROBLEM OF LAGRANGE
WITH ISOPERIMETRICAL SIDE CONDITIONS

1. Introduction. The central purpose of this paper is to develop a generalized Lindeberg theorem and apply it to the study of sufficient conditions for the parametric problem of Lagrange with isoperimETRICAL side conditions. This problem can be formulated descriptively as follows. Let K denote a class of curves of the form

$$x_i = x_i(t) \quad (t_1 \leq t \leq t_2; i = 1, 2, \dots, k),$$

on each of which the integral

$$J = \int_{t_1}^{t_2} f(x, x') dt$$

has a well-defined unique value. Let the class K' consist of all those curves of K which pass through two fixed points $(x_{11}, x_{12}, \dots, x_{1k})$ and $(x_{21}, x_{22}, \dots, x_{2k})$, which satisfy the set of differential equations

$$\mathcal{Q}_\beta(x, x') = 0 \quad (\beta = 1, 2, \dots, m < k),$$

and for which

$$I_\sigma = \int_{t_1}^{t_2} g_\sigma(x, x') dt = \ell_\sigma \quad (\sigma = 1, 2, \dots, h).$$

We seek to minimize the integral J in the class K' .

By means of the simple transformation

$$z_\sigma(t) = \int_{t_1}^t g_\sigma(x, x') dt,$$

this problem can be reduced to an equivalent parametric Lagrange problem. The new problem is to minimize the integral J in the class of curves of the form

$$x_1 = x_1(t), \quad z_\sigma = z_\sigma(t) \quad (t_1 \leq t \leq t_2),$$

which pass through the two fixed points $(x_{11}, x_{12}, \dots, x_{1k}, 0, 0, \dots, 0)$ and $(x_{21}, x_{22}, \dots, x_{2k}, \ell_1, \ell_2, \dots, \ell_n)$, and satisfy the set of differential equations

$$\mathcal{Q}_\beta(x, x') = 0, \quad g_\sigma(x, x') - z'_\sigma = 0.$$

If the curve $\bar{C}$: $x_1 = x_1(t), z_\sigma = z_\sigma(t), (t_1 \leq t \leq t_2)$, provides a relative minimum for the reduced problem, then the curve C : $x_1 = x_1(t), (t_1 \leq t \leq t_2)$, provides a relative minimum for the original problem on the sub-class of K' consisting of those curves of K' for which the Weierstrass construction is possible. However, in this paper, we shall show, by an argument involving a generalized Lindeberg theorem, that C provides a relative minimum for the original problem whenever $\bar{C}$ does so for the reduced problem. We are thus able to deduce sufficient conditions for the parametric isoperimetric Lagrange problem directly from those for the parametric Lagrange problem.

In addition to the application of the Lindeberg theorem mentioned above, we apply this theorem to deduce sufficient conditions for the parametric Lagrange problem from those for the non-parametric problem. To this effect we introduce the non-parametric Lagrange problem of minimizing the integral J in the class of curves of the form

$$x_1 = x_1(s) \quad (s_1 \leq s \leq s_2),$$

which satisfy the end conditions $s_1 = x_1(s_1) - x_{11} =$

$x_1(s_2) - x_{21} = 0$, and the set of differential equations

$$Q_\beta(x, x') = 0, \quad \sum_{i=1}^k x_i'^2 = 1.$$

We shall show that if $C: x_1 = x_1(s)$ provides a relative minimum for the non-parametric problem, then it also does so for the parametric problem.

The final application of the generalized Lindeberg theorem of this paper will be made to the development of two generalized Osgood theorems applicable to the parametric isoperimetric Lagrange problem.

In this paper we shall make use of Lebesgue integration, and the methods and results established by Tonelli [1]* and extended by McShane [2]. These methods are more direct and general than those employed by Lindeberg [3]. However, the neighborhoods used in this paper are more restricted than those ordinarily used in the Calculus of Variations. In the Calculus of Variations a curve C is said to be in the δ neighborhood of another curve C_0 if for every point on C there corresponds a point on C_0 such that the distance between the two points is less than δ . In this paper we shall say that a curve C is in the δ neighborhood of a curve C_0 if there is a simultaneous parameterization of the two curves C and C_0 such that the distance between corresponding points is less than δ . Such a neighborhood is termed by Tonelli an ordered neighborhood. It is significant to note that we shall make no use of the notion of a field.

2. Analytical Formulation. In order to simplify the notations as much as possible we shall denote points $(x_1, \dots, x_k)$

*Hereafter, numerals in square brackets will refer to the bibliography at the end of this paper.

in k dimensional x -space by x or x_1 . The symbols p, q, r, u, v , and w will denote unit vectors in k dimensional vector space. We shall omit summation signs and indicate summation by equating indices, e.g., $x_{11}x_{21} = \sum_{i=1}^k x_{1i}x_{2i}$.

By a continuous curve we shall mean an ordered set of points in x -space determined by a set of continuous functions $x_1(t)$, $(t_1 \leq t \leq t_2; i = 1, 2, \dots, k)$. The functions $x_1(t)$ may be simultaneously constant on an interval $t' \leq t \leq t''$. In particular $x_1(t)$ may be constant on the whole range so that the curve reduces to a point. Two continuous curves will be regarded as identical if and only if they consist of the same points in the same order, i.e., $C_1: x = x_1(t), (t_1 \leq t \leq t_2)$ is identical with $C_2: x = x_2(\theta), (\theta_1 \leq \theta \leq \theta_2)$ if and only if there is a correspondence between t and θ such that $x_1(t) = x_2(\theta)$ and $t'' \geq t'$ if and only if $\theta'' \geq \theta'$. In the preceding correspondence t is a monotone increasing function of θ . The correspondence, however, is not necessarily continuous or single-valued.

We define the distance between two continuous curves C_1 and C_2 as follows. Let π denote a simultaneous continuous parameterization of the two curves $C_1: x = x_1(t), (t_1 \leq t \leq t_2)$ and $C_2: x = x_2(t), (t_1 \leq t \leq t_2)$. Let d_π denote the least upper bound of $\|x_1(t) - x_2(t)\|$ for all t , the symbol $\|x\|$ denoting $\sqrt{x_1x_1}$. The distance $D = \|C_1, C_2\|$ between the two curves is defined to be the greatest lower bound of d_π for all simultaneous continuous parameterizations π . For further properties of the distance function the reader is referred to Fréchet [4].

As is well known, a continuous curve is rectifiable if and only if the functions defining it are of bounded variation. A continuous rectifiable curve can be given infinitely many absolutely continuous parameterizations. If C_1 and C_2 are

continuous and rectifiable, then the value of the distance $\|C_1, C_2\|$ obtained by admitting only absolutely continuous parameterizations is the same as that obtained by considering all continuous parameterizations.¹ A similar remark applies in case only one of the curves is rectifiable. We say that a sequence of continuous curves has a continuous limit curve C_0 in case the distance $\|C_n, C_0\|$ tends to zero with $1/n$. A continuous curve C is said to be in the δ neighborhood of a continuous curve C_0 , which we denote by $(C_0)_\delta$, in case $\|C, C_0\| < \delta$.

If $\{C_n\}$ is a sequence of continuous rectifiable curves which converges to a continuous rectifiable curve C_0 , and $\{C_n\}$ and C_0 are simultaneously parameterized, and the parameterizations $C_0: x = x_0(t)$, $(0 \leq t \leq 1)$ and $C_n: x = x_n(t)$, $(0 \leq t \leq 1)$ are so chosen that $x_0(t)$ and $x_n(t)$ are absolutely continuous, $\lim_{n \rightarrow \infty} x_n(t) = x_0(t)$ uniformly on $(0 \leq t \leq 1)$, and the sequence $\{x_n'(t)\}$ is uniformly bounded, then

$$(2:1) \quad \begin{aligned} \overline{\lim}_{n \rightarrow \infty} [J(C_n) - J(C_0)] &= \overline{\lim}_{n \rightarrow \infty} \int_0^1 E(x_0, x_0', x_n') dt, \\ \underline{\lim}_{n \rightarrow \infty} [J(C_n) - J(C_0)] &= \underline{\lim}_{n \rightarrow \infty} \int_0^1 E(x_0, x_0', x_n') dt, \end{aligned}$$

where for all t such that $x_0'(t)$ is undefined or $x_0'(t) = 0$ we replace $x_0'(t)$ by an arbitrary unit vector such that $x_0'(t)$ remains measurable. In particular we can replace $x_0'(t)$, wherever it is undefined or vanishes, by a constant unit vector.²

If a sequence of continuous rectifiable curves converges to a continuous rectifiable curve C_0 , and the sequence of lengths is uniformly bounded, then C_0 and a subsequence $\{C_n\}$ can be

¹McShane [2], p. 4.

²Ibid., p. 9.

parameterized simultaneously in such a way that the conditions mentioned immediately preceding the formulae (2:1) are satisfied.¹ In particular we can parameterize C_0 and a subsequence $\{C_n\}$ by setting $t = s/L(C_n)$, where $L(C_n)$ denotes the total length of C_n and s denotes the arc-length measured from the initial point of C_n to a generic point on it, and the conditions preceding the formulae (2:1) will be fulfilled.

Let A denote a bounded closed region of points in x -space. We shall assume that the functions $f(x, x')$, $g_\sigma(x, x')$, $Q_\beta(x, x')$ together with their first and second partial derivatives with respect to x' are continuous for x in A and x' arbitrary but not zero. We also require $f(x, x')$, $g_\sigma(x, x')$, and $Q_\beta(x, x')$ to be positively homogeneous of degree 1 in x' , i.e., $f(x, kx') = kf(x, x')$, ($k > 0$). A point (x, x') will be said to be admissible if x is in A and $x' \neq 0$. By an admissible curve we shall mean a continuous rectifiable curve in A . We shall consider only absolutely continuous parameterizations of admissible curves. On an admissible curve the integral J is well defined and is independent of the particular parameterization.²

The principal subscripts used in this paper and their ranges are the following:

$$\begin{aligned} i, j &= 1, 2, \dots, k, & \sigma &= 1, 2, \dots, h, \\ \beta &= 1, 2, \dots, m < k, & n &= 1, 2, \dots, \infty. \end{aligned}$$

The letter K will denote the class of all admissible curves which satisfy the set of differential equations $Q_\beta(x, x') = 0$ and the isoperimetrical side conditions $I_\sigma = \ell_\sigma$. The class K is said to

¹McShane [2], p. 10.

²Tonelli [1], I, p. 219.

be closed if every rectifiable limit curve of curves of K is also a curve of K . In general the class K will not be closed. The integral J is said to be lower semi-continuous with respect to the class K at a given curve C_0 of that class if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every C in K and in $(C_0)_\delta$ we have $J(C) - J(C_0) \geq -\varepsilon$. This is equivalent to the condition that for every sequence $\{C_n\}$ of curves of K such that $\lim_{n \rightarrow \infty} C_n = C_0$, $\lim_{n \rightarrow \infty} J(C_n) \geq J(C_0)$.

The formulae (2:1) can be generalized in the following manner. Let $\{C_n\}$ be a sequence of curves of K which converges to a curve C_0 of the class K . Let C_0 and the sequence $\{C_n\}$ be simultaneously parameterized in such a way that the conditions mentioned immediately preceding the formulae (2:1) are satisfied. Let $\lambda_\beta(x), \mu_\sigma$ be a set of multipliers such that $\lambda_\beta(x)$ are defined and continuous on C_0 and the μ_σ are constants. Let $E_H(x, x', y'; \lambda, \mu)$ be defined as the Weierstrass E function formed for $H(x, x'; \lambda, \mu) = f(x, x') + \lambda_\beta Q_\beta(x, x') + \mu_\sigma g_\sigma(x, x')$. Then

$$(2:2) \quad \begin{aligned} \overline{\lim}_{n \rightarrow \infty} [J(C_n) - J(C_0)] &= \overline{\lim}_{n \rightarrow \infty} \int_0^1 E_H(x_0, x_0', x_n'; \lambda(x_0), \mu) dt, \\ \underline{\lim}_{n \rightarrow \infty} [J(C_n) - J(C_0)] &= \underline{\lim}_{n \rightarrow \infty} \int_0^1 E_H(x_0, x_0', x_n'; \lambda(x_0), \mu) dt. \end{aligned}$$

The remarks following the formulae (2:1) are applicable here as well. The above formulae follow immediately from (2:1).

In treating sequences of curves of the class K of uniformly bounded lengths, we shall make the following agreement. By a simultaneous parameterization such that the formulae (2:2) are applicable on a subsequence we shall mean the particular simultaneous parameterization mentioned in the paragraph following the formulae (2:1).

3. The Lindeberg Theorem. In this section we shall develop a generalized Lindeberg theorem suitable for application to the parametric Lagrange problem with isoperimetrical side conditions.

LEMMA 3:1. For every x in A and every $\epsilon > 0$ there exists a $\delta > 0$ such that if (y, u) is an admissible element satisfying $\varphi_\beta(y, u) = 0$ and y is in $(x)_\delta$, then there exists a vector v such that $\varphi_\beta(x, v) = 0$, and u is in $(v)_\epsilon$.

Let us suppose that the lemma is false. Then there exists an x in A , an $\epsilon > 0$, and a sequence of admissible elements $\{(y_n, u_n)\}$ such that $\varphi_\beta(y_n, u_n) = 0$, y_n is in $(x)_{1/n}$, and $\|u_n - q\| \geq \epsilon$ for all n and all q satisfying $\varphi_\beta(x, q) = 0$. Select a subsequence such that $\{u_n\}$ converges to a vector v . Since $\varphi_\beta(x, x')$ is continuous, it follows that $\varphi_\beta(x, v) = 0$. There exists an N such that $\|u_n - v\| < \epsilon$ for all $n \geq N$. This is a contradiction, and hence the lemma is established.

LEMMA 3:2. Let $\{C_n\}$ be a sequence of curves of the class K . Let C_0 be a continuous limit curve of this sequence. Let there exist a set of multipliers $\lambda_\beta(x), \mu_\sigma$ and a vector function $r(x)$ such that $\lambda_\beta(x)$ and $r(x)$ are each defined on C_0 and the μ_σ are constants, and $E_H(x_0, r(x_0), q; \lambda(x_0), \mu) > 0$ for every element (x_0, q) such that $\varphi_\beta(x_0, q) = 0$ and $q \neq r(x_0)$. If $\lim_{n \rightarrow \infty} L(C_n) = +\infty$, then $\lim_{n \rightarrow \infty} J(C_n) = +\infty$.

Parameterize C_0 and the sequence $\{C_n\}$ such that the functions $x_n(t)$ are absolutely continuous and $\lim_{n \rightarrow \infty} x_n(t) = x_0(t)$ uniformly on $(0 \leq t \leq 1)$. By hypothesis

$$H(x_0, q; \lambda(x_0), \mu) - q_1 H_{x_1}'(x_0, r(x_0); \lambda(x_0), \mu) > 0$$

for every (x_0, q) such that $\varphi_\beta(x_0, q) = 0$ and $q \neq r(x_0)$. There exists an $\eta > 0$ such that

$$H(x_0, q; \lambda(x_0), \mu) - q_1 H_{x_1}(x_0, r(x_0); \lambda(x_0), \mu) \geq 2\eta > 0$$

for all q such that $\mathcal{Q}_\beta(x_0, q) = 0$ and $\|q - r(x_0)\| \geq 1$. It is easily seen that if $\|q - r(x_0)\| < 1$, then $q_1 r_1(x_0) > 1/2$. We define $M_1(x_0) \equiv H_{x_1}(x_0, r(x_0); \lambda(x_0), \mu) - \eta r_1(x_0)$. Hence,

$$H(x_0, q; \lambda(x_0), \mu) - q_1 M_1(x_0) > \eta/2$$

for all q such that $\mathcal{Q}_\beta(x_0, q) = 0$. There exists a $\rho > 0$ and an $\xi > 0$ such that

$$H(x, u; \lambda(x_0), \mu) - u_1 M_1(x_0) > \eta/4$$

for every x in $(x_0)_{3\rho}$ and every u in $(q)_\xi$, where q satisfies $\mathcal{Q}_\beta(x_0, q) = 0$. By Lemma 3:1 there exists a $\delta > 0$ such that if (x, v) is an admissible element satisfying $\mathcal{Q}_\beta(x, v) = 0$ and x is in $(x_0)_{3\delta}$, then there exists a vector q such that $\mathcal{Q}_\beta(x_0, q) = 0$ and v is in $(q)_\xi$. Let γ be the smaller of ρ and δ . Then

$$H(x, u; \lambda(x_0), \mu) - u_1 M_1(x_0) > \eta/4$$

for every x in $(x_0)_{3\gamma}$ and every u for which there exists a y in $(x_0)_{3\gamma}$ such that (y, u) is admissible and $\mathcal{Q}_\beta(y, u) = 0$.

To each point x_0 of C_0 we make correspond a neighborhood $(x_0)_\gamma$. Since the set of points C_0 is bounded and closed, there exists a finite subset T of C_0 such that the neighborhoods corresponding to points of T cover C_0 . We shall denote the points of T by y_0 . Let f be the smallest of the numbers corresponding to points y_0 in T . Subdivide the interval $[0, 1]$ into a finite number of sub-intervals $\Delta\tau \equiv [\alpha_\tau, \alpha_{\tau+},]$ such that the oscillation of C_0 on $\Delta\tau$ does not exceed f . There exists a point y_{τ_0} in T such that $x_0(\alpha_\tau)$ is in $(y_{\tau_0})_f$. Hence, for all t in $\Delta\tau$, $x_0(t)$ is in $(y_{\tau_0})_{2f}$. There exists an N such that $x_n(t)$ is in $(x_0(t))_f$ for all t and all $n \geq N$. Hence, for all t in $\Delta\tau$

and all $n \geq N$, $x_n(t)$ is in $(y_{r_0})_{3r}$. Define $u_n(t) \equiv x_n'(t)/\|x_n'(t)\|$ wherever $x_n'(t)$ is defined and $x_n'(t) \neq 0$, $M_1(t) \equiv M_1(y_{r_0})$ on Δr , and $\bar{\lambda}(t) = \lambda(y_{r_0})$ on Δr . Hence, for almost all t and all $n \geq N$

$$(3:1) \quad H(x_n(t), u_n(t); \bar{\lambda}(t), \mu) - u_{n_1}(t)M_1(t) > \eta_4.$$

Now,

$$J(C_n) = \int_0^1 [H(x_n(t), x_n'(t); \bar{\lambda}(t), \mu) - x_{n_1}'(t)M_1(t)]dt \\ + \int_0^1 x_{n_1}'(t)M_1(t)dt - \mu_\sigma \ell_\sigma.$$

Hence, except for a bounded term,

$$J(C_n) = \int_0^1 [H(x_n(t), x_n'(t); \bar{\lambda}(t), \mu) - x_{n_1}'(t)M_1(t)]dt.$$

Since $H(x, x'; \lambda, \mu)$ is positively homogeneous of degree 1 in x' ,

$$J(C_n) = \int_0^1 \|x_n'(t)\| [H(x_n(t), u_n(t); \bar{\lambda}(t), \mu) - u_{n_1}(t)M_1(t)]dt.$$

From (3:1) we have

$$J(C_n) > \frac{\eta}{4} L(C_n).$$

Hence, if $\lim_{n \rightarrow \infty} L(C_n) = +\infty$, then $\lim_{n \rightarrow \infty} J(C_n) = +\infty$.

THEOREM 3:1. Let C_0 be a curve of the class K . Let there exist a set of multipliers $\lambda_\beta(x)$, μ_σ and a vector function $r(x)$, where the $\lambda_\beta(x)$ and $r(x)$ are each defined and continuous on C_0 and the μ_σ are constants, such that $E_H(x_0, r(x_0), q; \lambda(x_0), \mu) > 0$ for all q such that $\phi_\beta(x_0, q) = 0$ and $q \neq r(x_0)$. Let $E_H(x_0(s), x_0'(s), q; \lambda(x_0(s)), \mu) \geq 0$ for almost all s and all q such that $\phi_\beta(x_0(s), q) = 0$, s being the arc-length measured on C_0 from the initial point to a generic point on it. Then $J(C)$ is

lower semi-continuous at C_0 with respect to the class K .

Suppose there exists a sequence $\{C_n\}$ of curves of the class K converging to C_0 , such that $\lim_{n \rightarrow \infty} J(C_n) = J(C_0) - \epsilon$. A subsequence $\{C_n\}$ can always be chosen such that $\lim_{n \rightarrow \infty} J(C_n) = J(C_0) - \epsilon$, and the sequence of lengths $\{L(C_n)\}$ also converges, say to L . For if $\lim_{n \rightarrow \infty} L(C_n) = +\infty$, then $\lim_{n \rightarrow \infty} J(C_n) = +\infty$, by Lemma 3:2. Parameterize C_0 and the sequence so that the formulae (2:2) are applicable on a subsequence. Let M designate the set of points of $[0, 1]$ where $x_0'(t)$ is undefined or $x_0'(t) = 0$. Define $u_0(t) \equiv x_0'(t)/\|x_0'(t)\|$ on the complement of M , and $u_0(t) \equiv r(x_0(t))$ on M . It is clear that the function $u_0(t)$ so defined is measurable. Define $u_n(t) \equiv x_n'(t)/\|x_n'(t)\|$ wherever $x_n'(t)$ exists and $x_n'(t) \neq 0$. Let E_n denote the set of t points on C_n at which $\mathcal{Q}_\beta(x_n, u_n) \neq 0$. The set E_n is of measure zero. Similarly, let E_0 denote the subset of the complement of M corresponding to the set of measure zero of s points for which $E_H(x_0(s), x_0'(s), q; \lambda(x_0(s)), \mu) < 0$ for at least one element $(x_0(s), q)$ for which $\mathcal{Q}_\beta(x_0(s), q) = 0$. Let $E = E_0 + \sum E_n$. The set E is of measure zero.¹ We shall show that $\lim_{n \rightarrow \infty} E_H(x_0(t), u_0(t), u_n(t); \lambda(x_0(t)), \mu) \geq 0$ except possibly on the set E . Suppose that the above were not true. Then there exists a point t not in E , a $\rho > 0$, and a subsequence $\{C_n\}$ such that $E_H(x_0(t), u_0(t), u_n(t); \lambda(x_0(t)), \mu) \leq -\rho/2$ for all curves C_n of this subsequence. Now, $E_H(x_0(t), u_0(t), q; \lambda(x_0(t)), \mu) \geq 0$ for all $(x_0(t), q)$ satisfying $\mathcal{Q}_\beta = 0$. There exists a $\theta > 0$ such that $E_H(x_0(t), u_0(t), v; \lambda(x_0(t)), \mu) \geq -\rho/4$ for all v such that there exists a q for which $\mathcal{Q}_\beta(x_0(t), q) = 0$ and $\|v - q\| \leq \theta$. By Lemma 3:1 there exists a $\delta > 0$ such that if x is in $(x_0(t))_\delta$,

¹Carathéodory, Vorlesungen Über Reelle Funktionen, p. 560, Satz 2.

(x, u) is admissible, and $\mathcal{Q}_\beta(x, u) = 0$, then there exists a q such that $\mathcal{Q}_\beta(x_0(t), q) = 0$ and $\|u - q\| \leq \theta$. There exists an N such that $x_n(t)$ is in $(x_0(t))_\theta$ for all $n \geq N$. Hence

$$E_H(x_0(t), u_0(t), u_n(t); \lambda(x_0(t)), \mu) \geq -\theta^2/4 \text{ for all } n \geq N.$$

Clearly, this is a contradiction, and hence

$\lim_{n \rightarrow \infty} E_H(x_0(t), u_0(t), u_n(t); \lambda(x_0(t)), \mu) \geq 0$ except possibly on the set E . Applying the formulae (2:2),

$$\begin{aligned} \lim_{n \rightarrow \infty} [J(C_n) - J(C_0)] &= \lim_{n \rightarrow \infty} \int_0^1 E_H(x_0(t), u_0(t), x_n'(t); \lambda(x_0(t)), \mu) dt \\ &= \lim_{n \rightarrow \infty} \int_0^1 \|x_n'(t)\| E_H(x_0(t), u_0(t), u_n(t); \lambda(x_0(t)), \mu) dt \\ &= \lim_{n \rightarrow \infty} L(C_n) \int_0^1 E_H(x_0(t), u_0(t), u_n(t); \lambda(x_0(t)), \mu) dt \\ &= L \lim_{n \rightarrow \infty} \int_0^1 E_H(x_0(t), u_0(t), u_n(t); \lambda(x_0(t)), \mu) dt. \end{aligned}$$

By a general result of Lebesgue integration,¹

$$\begin{aligned} \lim_{n \rightarrow \infty} [J(C_n) - J(C_0)] &\geq L \int_0^1 \lim_{n \rightarrow \infty} E_H(x_0(t), u_0(t), u_n(t); \lambda(x_0(t)), \mu) dt \\ &\geq 0. \end{aligned}$$

Hence the theorem is established.

The above theorem is still valid if we allow the vector function $r(x)$ to be completely arbitrary. However, the demonstration in this case is more complicated than the one given above.

THEOREM 3:2. LINDBERG THEOREM GENERALIZED. Let C_0 be a curve of the class K . Let there exist a set of multipliers $\lambda_\beta(x)$, μ_σ and a vector function $r(x)$, where the $\lambda_\beta(x)$ and $r(x)$ are each defined and continuous on C_0 and the μ_σ are constants, such that $E_H(x_0, r(x_0), q; \lambda(x_0), \mu) > 0$ for all q such that

¹Carathéodory, Vorlesungen über Reelle Funktionen, p. 444, satz 16.

$\mathcal{Q}_\beta(x_0, q) = 0$ and $q \neq r(x_0)$. Let $E_H(x_0(s), x_0'(s), q; \lambda(x_0(s)), \mu) > 0$ for almost all s and all q such that $\mathcal{Q}_\beta(x_0(s), q) = 0$ and $q \neq x_0'(s)/\|x_0'(s)\|$, s being the arc-length measured on C_0 from the initial point to a generic point on it. Then for every $\delta > 0$ there exists a $\rho > 0$ and an $\varepsilon > 0$ such that $J(C) - J(C_0) > \varepsilon$ for all curves C of the class K in $(C_0)_\rho$ for which $L(C) - L(C_0) \geq \delta$.

Let us suppose that the theorem is false. Then there exists a $\delta > 0$ and a sequence $\{C_n\}$ of curves of the class K such that C_n is in $(C_0)_{1/n}$, $L(C_n) - L(C_0) \geq \delta$, and $J(C_n) - J(C_0) \leq 1/n$. By Theorem 3:1 we have $\lim_{n \rightarrow \infty} J(C_n) = J(C_0)$. By Lemma 3:2 the numbers $L(C_n)$ are uniformly bounded. Select a subsequence such that $\lim_{n \rightarrow \infty} L(C_n) = L$. Parameterize C_0 and a subsequence so that the formulae (2:2) are applicable. Let M designate the set of points of $[0, 1]$ where $x_0'(t)$ is undefined or $x_0'(t) = 0$. The set M is measurable. Define $u_0(t) \equiv x_0'(t)/\|x_0'(t)\|$ on the complement of M and $u_0(t) \equiv r(x_0(t))$ on M . Define $u_n(t) \equiv x_n'(t)/\|x_n'(t)\|$ wherever $x_n'(t)$ exists and $x_n'(t) \neq 0$. Now, the sequence of functions $\{(1 - u_{0i}(t)u_{ni}(t))\}$ cannot converge approximately to zero. For if it did, then

$$\lim_{n \rightarrow \infty} \int_0^1 [1 - u_{0i}(t)u_{ni}(t)] dt = 0.$$

Applying the formulae (2:1) we have

$$\begin{aligned} \lim_{n \rightarrow \infty} [L(C_n) - L(C_0)] &= \lim_{n \rightarrow \infty} \int_0^1 E_\ell(x_0(t), u_0(t), x_n'(t)) dt \\ &= \lim_{n \rightarrow \infty} \int_0^1 \|x_n'(t)\| [1 - u_{0i}(t)u_{ni}(t)] dt \\ &= \lim_{n \rightarrow \infty} L(C_n) \int_0^1 [1 - u_{0i}(t)u_{ni}(t)] dt \\ &= 0. \end{aligned}$$

Clearly, then, there exists an $\eta > 0$ and a subsequence $\{C_n\}$ such

that the measure of the set S_{1n} where $\|u_0(t) - u_n(t)\| \geq 8\eta$ must be greater than 3η . There exists a set S_2 such that $m(S_2) \geq 1 - \eta$ and $u_0(t)$ is continuous on S_2 .¹ Let E_n denote the set of measure zero of t points on C_n at which $\mathcal{Q}_\beta(x_n, u_n) \neq 0$. Let E_0 denote the subset of the complement of M corresponding to the set of measure zero of s points on C_0 for which $E_H(x_0(s), x_0'(s), q; \lambda(x_0(s)), \mu) \leq 0$ for at least one element $(x_0(s), q)$ such that $\mathcal{Q}_\beta(x_0(s), q) = 0$ and $q \neq x_0'(s)/\|x_0'(s)\|$. Define $E \equiv E_0 + \sum E_n$. The set E has measure zero. There exists a closed set S_3 , contained in S_2 , which is free of points of E and has measure not less than $1 - 2\eta$. Define $S_n \equiv S_{1n} \cdot S_3$. Then $m(S_n) \geq \eta$, and $\|u_0(t) - u_n(t)\| \geq 8\eta$ on S_n . Now, for every t in S_3 , $E_H(x_0(t), u_0(t), q; \lambda(x_0(t)), \mu) > 0$ for all q such that $\mathcal{Q}_\beta(x_0(t), q) = 0$ and $q \neq u_0(t)$. Select $m > 3/\eta$. Then there exists a $\nu > 0$ such that for every t in S_3 , $E_H(x_0(t), u_0(t), q; \lambda(x_0(t)), \mu) \geq (m + 2)\nu$ for all q such that $\mathcal{Q}_\beta(x_0(t), q) = 0$ and $\|q - u_0(t)\| \geq 2\eta$; and $E_H(x_0(t), u_0(t), q; \lambda(x_0(t)), \mu) \geq 0$ for all q such that $\mathcal{Q}_\beta(x_0(t), q) = 0$. There exists a $\theta > 0$ which can be selected so as not to exceed 2η , such that for every t in S_3 , $E_H(x_0(t), u_0(t), v; \lambda(x_0(t)), \mu) \geq (m + 1)\nu$ for all v for which there exists a q such that $\mathcal{Q}_\beta(x_0(t), q) = 0$, $\|q - u_0(t)\| \geq 2\eta$, and $\|q - v\| \leq \theta$; and $E_H(x_0(t), u_0(t), v; \lambda(x_0(t)), \mu) \geq -\nu$ for all v for which there exists a q such that $\mathcal{Q}_\beta(x_0(t), q) = 0$ and $\|q - v\| \leq \theta$. By Lemma 3:1, for every t there exists a $\xi > 0$ such that if x is in $(x_0(t))_{3\xi}$, (x, u) is admissible, and $\mathcal{Q}_\beta(x, u) = 0$, then there exists a q such that $\mathcal{Q}_\beta(x_0(t), q) = 0$ and $\|u - q\| \leq \theta$. There exists a $\rho_1 > 0$ such that $\|x_0(t) - x_0(t')\| \leq 2\xi$ for every

¹Lebesgue, Comptes Rendus de l'Academie des Sciences, 137 (1903), pp. 1228-30.

t and t' such that $|t - t'| \leq 2\rho_1$. There exists a $\rho_2 > 0$ such that $\|u_0(t) - u_0(t')\| \leq \eta$ for every two points t and t' in S_3 for which $|t - t'| \leq 2\rho_2$. There exists a $\rho_3 > 0$ such that for every two points t and t' in S_3 for which $|t - t'| \leq 2\rho_3$, $E_H(x_0(t), u_0(t), v; \lambda(x_0(t)), \mu) \geq m\nu$ for all v for which there exists a q such that $\mathcal{Q}_\beta(x_0(t'), q) = 0$, $\|q - u_0(t')\| \geq 2\eta$, and $\|q - v\| \leq \theta$; and $E_H(x_0(t), u_0(t), v; \lambda(x_0(t)), \mu) \geq -2\nu$ for all v for which there exists a q such that $\mathcal{Q}_\beta(x_0(t'), q) = 0$ and $\|q - v\| \leq \theta$. Let γ be the smallest of the ρ_1 , ρ_2 , and ρ_3 . Then, for every t' in S_3 we have $\|x_0(t) - x_0(t')\| \leq 2\gamma$, $\|u_0(t) - u_0(t')\| \leq \eta$, and $E_H(x_0(t), u_0(t), v; \lambda(x_0(t)), \mu) \geq m\nu$ for all t in S_3 for which $|t - t'| \leq 2\gamma$ and for all v for which there exists a q such that $\mathcal{Q}_\beta(x_0(t'), q) = 0$, $\|q - u_0(t')\| \geq 2\eta$, and $\|q - v\| \leq \theta$; and $E_H(x_0(t), u_0(t), v; \lambda(x_0(t)), \mu) \geq -2\nu$ for all v for which there exists a q such that $\mathcal{Q}_\beta(x_0(t'), q) = 0$ and $\|q - v\| \leq \theta$.

To each point t of S_3 we make correspond a closed neighborhood $(t)_\gamma$. By the Heine-Borel theorem there exists a finite subset T of S_3 such that the neighborhoods corresponding to points of T cover S_3 . Let δ be the smallest of the numbers γ corresponding to points of T . Subdivide the interval $[0, 1]$ into N_1 subintervals $\Delta\tau = [\alpha_\tau, \alpha_{\tau+1}]$ each of length $\delta = 1/N_1$. There exists a subinterval $\Delta\bar{\tau} = [\bar{\alpha}_\tau, \bar{\alpha}_{\tau+1}]$ such that $m(S_n \cdot \Delta\bar{\tau}) \geq \nu N_1$. There exists a $\bar{t}$ in T such that $|\bar{\alpha}_\tau - \bar{t}| \leq \delta$. Hence, $|t - \bar{t}| \leq 2\delta$ for all t in $\Delta\bar{\tau}$. For all t in $S_3 \cdot \Delta\bar{\tau}$ we have $\|x(t) - x(\bar{t})\| \leq 2\delta$, $\|u_0(t) - u_0(\bar{t})\| \leq \eta$, and $E_H(x_0(t), u_0(t), v; \lambda(x_0(t)), \mu) \geq m\nu$ for all v for which there exists a q such that $\mathcal{Q}_\beta(x_0(\bar{t}), q) = 0$, $\|q - u_0(\bar{t})\| \geq 2\eta$, and $\|q - v\| \leq \theta$; and $E_H(x_0(t), u_0(t), v; \lambda(x_0(t)), \mu) \geq -2\nu$ for all v such that there exists a q such that $\mathcal{Q}_\beta(x_0(\bar{t}), q) = 0$ and $\|q - v\| \leq \theta$. There exists an N such

that $\|x_n(t) - x_0(t)\| \leq \frac{1}{3}$ for all $n \geq N$. Hence, $\|x_n(t) - x_n(\bar{t})\| \leq \frac{1}{3}$ for all t in $S_3 \cdot \Delta \bar{t}$ and $n \geq N$. Now, for all t in S_n , $\|u_n(t) - u_0(t)\| \geq 8\eta$. Hence, for all t in $S_n \cdot \Delta \bar{t}$, $\|u_n(t) - u_0(\bar{t})\| \geq 6\eta$. For all t in $S_3 \cdot \Delta \bar{t}$ and all $n \geq N$ there exists a q such that $\mathcal{Q}_\beta(x_0(\bar{t}), q) = 0$ and $\|u_n(t) - q\| \leq \theta$. For all t in $S_n \cdot \Delta \bar{t}$ and all $n \geq N$ there exists a q such that $\mathcal{Q}_\beta(x_0(\bar{t}), q) = 0$, $\|q - u_0(\bar{t})\| \geq 2\eta$, and $\|u_n(t) - q\| \leq \theta$. Hence, $E_H(x_0(t), u_0(t), u_n(t); \lambda(x_0(t)), \mu) \geq m\nu$ for all t in $S_n \cdot \Delta \bar{t}$ and all $n \geq N$; and for all t in $S_3 \cdot \Delta \bar{t}$ and all $n \geq N$ we have $E_H(x_0(t), u_0(t), u_n(t); \lambda(x_0(t)), \mu) \geq -2\nu$. Applying the formulae (2:2)

$$\begin{aligned} \lim_{n \rightarrow \infty} [J(C_n) - J(C_0)] &= \lim_{n \rightarrow \infty} \int_0^1 E_H(x_0(t), u_0(t), x_n'(t); \lambda(x_0(t)), \mu) dt \\ &= \lim_{n \rightarrow \infty} \int_0^{\bar{t}} E_H(x_0(t), u_0(t), x_n'(t); \lambda(x_0(t)), \mu) dt \\ &\quad + \lim_{n \rightarrow \infty} \int_{\bar{t}}^1 E_H(x_0(t), u_0(t), x_n'(t); \lambda(x_0(t)), \mu) dt \\ &\quad + \lim_{n \rightarrow \infty} \int_{\Delta \bar{t}} E_H(x_0(t), u_0(t), x_n'(t); \lambda(x_0(t)), \mu) dt. \end{aligned}$$

By Theorem 3:1,

$$\lim_{n \rightarrow \infty} \int_0^{\bar{t}} E_H(x_0(t), u_0(t), x_n'(t); \lambda(x_0(t)), \mu) dt \geq 0,$$

and similarly for the second integral on the right-hand side of the above. Hence,

$$\begin{aligned} \lim_{n \rightarrow \infty} [J(C_n) - J(C_0)] &\geq \lim_{n \rightarrow \infty} \int_{\Delta \bar{t}} E_H(x_0(t), u_0(t), x_n'(t); \lambda(x_0(t)), \mu) dt \\ &\geq \lim_{n \rightarrow \infty} \int_{\Delta \bar{t}} \|x_n'(t)\| E_H(x_0(t), u_0(t), u_n(t); \lambda(x_0(t)), \mu) dt \\ &\geq \lim_{n \rightarrow \infty} L(C_n) \int_{\Delta \bar{t}} E_H(x_0(t), u_0(t), u_n(t); \lambda(x_0(t)), \mu) dt \\ &\geq L \left\{ \lim_{n \rightarrow \infty} \int_{S_n \cdot \Delta \bar{t}} E_H(x_0(t), u_0(t), u_n(t); \lambda(x_0(t)), \mu) dt \right. \\ &\quad \left. + \lim_{n \rightarrow \infty} \int_{(S_3 - S_n) \cdot \Delta \bar{t}} E_H(x_0(t), u_0(t), u_n(t); \lambda(x_0(t)), \mu) dt \right\} \end{aligned}$$

$$+ \lim_{n \rightarrow \infty} \int_{CS_3, \Delta \bar{e}} E_H(x_0(t), u_0(t), u_n(t); \lambda(x_0(t)), \mu) dt \Big\}$$

By means of the proof of Theorem 3:1,

$$\lim_{n \rightarrow \infty} \int_{CS_3, \Delta \bar{e}} E_H(x_0(t), u_0(t), u_n(t); \lambda(x_0(t)), \mu) dt \geq 0.$$

Hence,

$$\begin{aligned} \lim_{n \rightarrow \infty} [J(C_n) - J(C_0)] &\geq L(m\nu \frac{\eta}{N_1} - 2\nu \frac{1}{N_1}) \\ &\geq \frac{L\nu}{N_1} (m\eta - 2) > \frac{L\nu}{N_1} > 0. \end{aligned}$$

This is the desired contradiction, and the theorem is established.

The above theorem can be established under the weaker assumption that $r(x)$ is a completely arbitrary vector function defined on C_0 . However the demonstration under the assumption is more complicated than the one given above.

4. General Theorems on Relative Minima. In this section we shall show how to deduce sufficient conditions for the parametric problem of Lagrange with isoperimetrical side conditions from those for the parametric problem without isoperimetrical side conditions. We shall further show how to obtain sufficient conditions for the parametric problem of Lagrange from those for the non-parametric problem.

LEMMA 4:1. Let $\{C_n\}$ be a sequence of admissible curves converging to an admissible curve C_0 such that $\lim_{n \rightarrow \infty} L(C_n) = L(C_0)$. Let C_0 and $\{C_n\}$ be simultaneously parameterized by setting $t = s/L(C)$. Let $g(x, x')$ satisfy the same general hypotheses as $f(x, x')$. Define $z(t) = \int_0^t g(x, x') dt$. Then $\lim_{n \rightarrow \infty} z_n(t) = z_0(t)$ uniformly.

Since $g(x, x')$ is continuous, $g(x, u)$ is bounded. Since $\{x_n'(t)\}$ is uniformly bounded and $g(x_n, x_n') = \|x_n'\| g(x_n, u_n)$, it

follows that $g(x_n, x_n')$ are uniformly bounded. Hence, the sequence of functions $\{z_n(t)\}$ is equi-continuous. Now, the sequence $\{z_n(t)\}$ converges to $z_0(t)$.¹ Hence, the convergence is uniform,² and the lemma is established.

Let us now consider the parametric problem of Lagrange with isoperimetrical side conditions. As in the Introduction, we reduce this problem to a parametric Lagrange problem without isoperimetrical side conditions. For this reduced problem we define the following terms. An element (x, z, x', z') will be said to be admissible in enlarged space if (x, x') is admissible. By the notation $\bar{C}$ we shall understand a curve in enlarged space obtained from the curve $C: x_1 = x_1(t)$, $(t_1 \leq t \leq t_2)$ by the introduction of the auxiliary dependent variables

$$z_\sigma(t) = \int_{t_1}^t g_\sigma(x, x') dt.$$

A curve $\bar{C}: x_1 = x_1(t)$, $z_\sigma = z_\sigma(t)$, $(t_1 \leq t \leq t_2)$ will be said to be admissible in enlarged space if the curve $C: x_1 = x_1(t)$, $(t_1 \leq t \leq t_2)$ is admissible. The reduced problem is to minimize the integral J in the class $\bar{K}$ of admissible curves in enlarged space passing through the two fixed points $(x_{11}, 0)$ and (x_{21}, l_σ) , and satisfying the set of differential equations $\mathcal{Q}_\beta(x, x') = 0$ and $g_\sigma(x, x') - z'_\sigma = 0$ almost everywhere.

LEMMA 4:2. Let C_0 be an admissible curve. For every $\epsilon > 0$ there exists a $\rho > 0$ and a $\delta > 0$ such that if C is admissible and in $(C_0)_\rho$, and $\bar{C}$ is not in $(\bar{C}_0)_\epsilon$, then $L(C) - L(C_0) > \delta$.

¹McShane [2], p. 37.

²Graves, mimeographed notes on The Theory of Functions of a Real Variable (1931), p. 75.

Hahn, Reelle Funktionen, part 1 (1932), pp. 229, 224, 216.

Let us suppose that the lemma is false. Then there exists an $\varepsilon > 0$ and a sequence $\{C_n\}$ of admissible curves such that C_n is in $(C_0)_{1/n}$, $\bar{C}_n$ is not in $(\bar{C}_0)_\varepsilon$, and $L(C_n) - L(C_0) \leq 1/n$.

Since the length function is lower semi-continuous,

$\lim_{n \rightarrow \infty} L(C_n) = L(C_0)$. Parameterize C_0 and the sequence $\{C_n\}$ by setting $t = s/L(C)$. By Lemma 4:1, $\lim_{n \rightarrow \infty} z_{n\sigma}(t) = z_{0\sigma}(t)$ uniformly, and from the particular parameterization, $\lim_{n \rightarrow \infty} x_n(t) = x_0(t)$ uniformly.¹ Hence we have reached a contradiction, and the lemma is established.

THEOREM 4:1. Let $C_0: x_1 = x_1(t), (t_1 \leq t \leq t_2)$ be a curve of the class K which passes through the two fixed points x_{11} and x_{21} . Let there exist a set of multipliers $\lambda_\beta(x), \mu_\sigma$ and a vector function $r(x)$, where the $\lambda_\beta(x)$ and $r(x)$ are each defined and continuous on C_0 and the μ_σ are constants, such that $E_H(x_0, r(x_0), q; \lambda(x_0), \mu) > 0$ for all q such that $\mathcal{Q}_\beta(x_0, q) = 0$ and $q \neq r(x_0)$. Let $E_H(x_0(s), x_0'(s), q; \lambda(x_0(s)), \mu) > 0$ for almost all s and all q such that $\mathcal{Q}_\beta(x_0(s), q) = 0$ and $q \neq x_0'(s)/\|x_0'(s)\|$, s being the arc-length measured on C_0 from its initial point to a generic point on it. Let there exist a neighborhood of $\bar{C}_0$ such that $\bar{C}_0$ minimizes the integral J in the class of all the curves of $\bar{K}$ lying in this neighborhood. Then there exists a neighborhood of C_0 such that C_0 minimizes the integral J in the class of all the curves of K which lie in this neighborhood of C_0 and pass through the two fixed points x_{11} and x_{21} .

It is understood that the neighborhoods referred to in the theorem are of the restricted type mentioned in the Introduction.

¹McShane, [2], p. 36.

There exists a neighborhood of C_0 such that C_0 minimizes the integral J in the class of all the curves C of K which lie in this neighborhood, pass through the two fixed points x_{11} and x_{21} , and are such that $\bar{C}$ is in the neighborhood of $\bar{C}_0$ prescribed in the theorem. By Lemma 4:2 there exists a $\delta > 0$ and a neighborhood of C_0 such that if C is an admissible curve lying in this neighborhood and $\bar{C}$ is not in the neighborhood prescribed in the theorem, then $L(C) - L(C_0) > \delta$. By Theorem 3:2 there exists a neighborhood of C_0 and an $\epsilon > 0$ such that for every curve C of K in this neighborhood for which $L(C) - L(C_0) > \delta$ we have $J(C) - J(C_0) > \epsilon$. Select the smallest of the three neighborhoods of C_0 mentioned above. This neighborhood is effective in the conclusion of the theorem.

It is clear, then, that we can derive sufficient conditions for the parametric Lagrange problem with isoperimetrical side conditions directly from those for the parametric problem.

Let us now consider the parametric problem of Lagrange. As in the Introduction we introduce the non-parametric problem of minimizing the integral J in the class $\bar{K}_1$ of admissible curves which satisfy the end conditions $s_1 = x_1(s_1) - x_{11} = x_1(s_2) - x_{21} = 0$, and the set of differential equations $\mathcal{Q}_\beta(x, x') = 0$ and $x_1'x_1' = 1$. The class K_1 shall denote the totality of admissible curves satisfying the set of differential equations $\mathcal{Q}_\beta(x, x') = 0$. Let us define $E_P(x, x', y'; \lambda)$ as the Weierstrass E-function formed for $F(x, x'; \lambda) \equiv f(x, x') + \lambda_\beta \mathcal{Q}_\beta(x, x')$. It is plain that all the results of Section 3 are valid when the isoperimetric conditions are absent.

LEMMA 4.3. Let $C_0: x = x_0(s)$ be an admissible curve parameterized by means of its arc-length. For every positive ϵ there exist positive numbers ρ and δ such that if $C: x = x(s)$ is an admissible curve similarly parameterized, C is in $(C_0)_\rho$, and there exists an s such that $x(s)$ is not in $(x_0(s))_\epsilon$, then $L(C) - L(C_0) > \delta$.

Let us assume that the lemma is false. Then there exists a positive number ϵ and a sequence $\{C_n\}$ of admissible curves such that C_n is in $(C_0)_{1/n}$, $L(C_n) - L(C_0) \leq 1/n$, and for every n there exists an s_n such that $x_n(s_n)$ is not in $(x_0(s_n))_\epsilon$. Since the length function is lower semi-continuous, $\lim_{n \rightarrow \infty} L(C_n) = L(C_0)$. Parameterize C_0 and the sequence $\{C_n\}$

$$C_0: x = \bar{x}_0(t),$$

$$C_n: x = \bar{x}_n(t),$$

by setting $t = s/L(C_n)$, where s is the arc-length measured along C_n from its initial point to a generic point on it. Under this simultaneous parameterization $\lim_{n \rightarrow \infty} \bar{x}_n(t) = \bar{x}_0(t)$ uniformly on $(0 \leq t \leq 1)$. Now we have two sequences $\{\bar{x}_n(t_n)\}$ and $\{\bar{x}_0(\bar{t}_n)\}$, where $t_n = s_n/L(C_n)$ and $\bar{t}_n = s_n/L(C_0)$, such that

$\|\bar{x}_n(t_n) - \bar{x}_0(\bar{t}_n)\| > \epsilon$. Select a subsequence such that

$\lim_{n \rightarrow \infty} t_n = \bar{t}$. Then $\lim_{n \rightarrow \infty} s_n = \bar{s} = L(C_0)\bar{t}$ and $\lim_{n \rightarrow \infty} \bar{t}_n = \bar{t}$. Since the sequence $\{\bar{x}_n(t)\}$ is equi-continuous, there exists an N_1 such that

$$\|x_n(t_n) - x_n(\bar{t})\| < \epsilon/3 \text{ and } \|x_0(\bar{t}_n) - x_0(\bar{t})\| < \epsilon/3 \text{ for all } n \geq N_1.$$

There exists an N_2 such that $\|x_n(\bar{t}) - x_0(\bar{t})\| < \epsilon/3$ for all $n \geq N_2$.

Select N to be the greater of N_1 and N_2 . Then $\|x_n(t_n) - x_0(\bar{t}_n)\| \leq \|x_n(t_n) - x_n(\bar{t})\| + \|x_n(\bar{t}) - x_0(\bar{t})\| + \|x_0(\bar{t}_n) - x_0(\bar{t})\| < \epsilon$ for $n \geq N$. This is a desired contradiction, and hence the lemma is established.

THEOREM 4:2. Let $C_0: x_1 = x_{01}(s)$, ($s_1 \leq s \leq s_2$) be a curve of the class $\bar{K}_1$. Let there exist a set of multipliers $\lambda_\beta(x)$ and a vector function $r(x)$, where the $\lambda_\beta(x)$ and $r(x)$ are each defined and continuous on C_0 , such that $E_F(x_0, r(x_0), q; \lambda(x_0)) > 0$ for all q such that $\varphi_\beta(x_0, q) = 0$ and $q \neq r(x_0)$. Let $E_F(x_0(s), x_0'(s), q; \lambda(x_0(s))) > 0$ for almost all s and all q such that $\varphi_\beta(x_0(s), q) = 0$ and $q \neq x_0'(s)/\|x_0'(s)\|$. Let there exist a neighborhood of C_0 in non-parametric space such that C_0 minimizes the integral J in the class of all the curves of $\bar{K}_1$ which lie in this neighborhood. Then there exists a neighborhood of C_0 in parametric space such that C_0 minimizes the integral J in the class of all the curves of K_1 which lie in this neighborhood and pass through the two fixed points x_{11} and x_{21} .

The neighborhoods mentioned in the theorem are those of the restricted type mentioned in the Introduction.

There exists a neighborhood of C_0 in parametric space such that C_0 minimizes the integral J in the class of all the curves of K_1 which lie in this neighborhood, pass through the two fixed points x_{11} and x_{21} , and are such that (s, x) is in the neighborhood of (s, x_0) prescribed in the theorem. By Lemma 4:3 there exists a neighborhood of C_0 in parametric space and a $\delta > 0$ such that if C is an admissible curve lying in this neighborhood and the point (s, x) on C is not in the neighborhood of (s, x_0) prescribed in the theorem, then $L(C) - L(C_0) > \delta$. By Theorem 3:2 for the parametric problem, there exists a neighborhood of C_0 in parametric space and an $\epsilon > 0$ such that if C is in K_1 and in this neighborhood of C_0 , and $L(C) - L(C_0) > \delta$, then $J(C) - J(C_0) > \epsilon$. Select the smallest of the three neighborhoods of C_0 in parametric space mentioned above. This neighborhood is effective in the theorem.

It is seen that sufficient conditions for the parametric problem can be obtained directly from those for the non-parametric problem. Further, we can obtain sufficient conditions for the parametric isoperimetric problem of Lagrange directly from those for the non-parametric Lagrange problem. In the next section we shall derive a set of sufficient conditions for the parametric isoperimetric Lagrange problem.

5. Sufficient Conditions. Throughout this section we shall assume that $f(x, x')$, $g_\sigma(x, x')$, and $Q_\beta(x, x')$ are of class C''' for x in A and x' arbitrary and that the matrix $\|Q_{\beta x_i}\|$ is of rank m for all admissible (x, x') . By an admissible curve $C: x_1 = x_1(t), (t_1 \leq t \leq t_2)$ we shall now mean a curve in A which consists of at most a finite number of subarcs of class C' such that at each point of C the inequality $\|x'\| \neq 0$ holds. We shall now introduce the non-parametric Lagrange problem of minimizing the integral J in the class of curves of the form

$$x_1 = x_1(s), \quad z_\sigma = z_\sigma(s), \quad (s_1 \leq s \leq s_2),$$

which satisfy the end conditions $s_1 = x_1(s_1) - x_{11} = x_1(s_2) - x_{21} = z_\sigma(s_1) = z_\sigma(s_2) - \ell_\sigma = 0$, and the set of differential equations $Q_\beta(x, x') = 0, g_\sigma(x, x') - z'_\sigma = 0, x_1'x_1' + z'_\sigma z'_\sigma = 1$. An element (s, x, z, x', z') will be said to be admissible in non-parametric space if (x, x') is admissible and $s \geq 0$. A curve $\bar{C}: x_1 = x_1(s), z_\sigma = z_\sigma(s), (s_1 \leq s \leq s_2)$ will be said to be admissible in non-parametric space if the usual continuity assumptions are fulfilled and the curve $C: x_1 = x_1(s)$ is admissible. It is easily seen that the matrix

$$\begin{vmatrix} Q_{\beta x'_i} & 0 \\ g_{\sigma x'_1} & -I \\ x'_1 & z'_\sigma \end{vmatrix}$$

is of rank $m + h + 1$ for all admissible elements (s, x, z, x', z') in non-parametric space satisfying the equations $Q_{\beta}(x, x') = 0$ and $g_{\sigma}(x, x') - z'_{\sigma} = 0$. The symbol I denotes the identity matrix

$$\begin{vmatrix} 1 & & & & \\ & 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & 1 \end{vmatrix}$$

which is in this case an $h \times h$ matrix.

Let $E: x_1 = x_1(t)$, $(t_1 \leq t \leq t_2)$ be a curve of the class K without corners which passes through the two fixed points x_{11} and x_{21} and lies interior to the region A . Let there exist a set of multipliers $\lambda_{\beta}(x)$, μ_{σ} such that the $\lambda_{\beta}(x)$ are defined and are of class C^1 on E and the μ_{σ} are constants. As in Section 2 we define $H(x, x'; \lambda, \mu) = f(x, x') + \lambda_{\beta} Q_{\beta}(x, x') + \mu_{\sigma} g_{\sigma}(x, x')$. We shall say that the arc E satisfies the condition I if it satisfies the set of differential equations

$$\frac{d}{dt} H_{x'_1} = H_{x_1}.$$

If along E the quadratic form

$$r_1 H_{x'_1 x'_j}(x, x'; \lambda(x), \mu) r_j > 0$$

for all $r_1 \neq x'_1 / \|x'_1\|$ satisfying

$$Q_{\beta x'_1}(x, x') r_1 = 0,$$

then we shall say that the arc E satisfies the condition III'.

We shall say that the arc E satisfies the condition II' if along E

$$E_H(x, x', q; \lambda(x), \mu) > 0$$

for all q such that $Q_\beta(x, q) = 0$ and $q \neq x'/\|x'\|$.

The problem of the second variation associated with the parametric problem is that of minimizing the integral

$$J_2(\eta) = \int_{t_1}^{t_2} 2\omega(\eta, \eta') dt$$

in the class of curves of the form

$$\eta_1 = \eta_1(t), \quad f_\sigma = f_\sigma(t) \quad (t_1 \leq t \leq t_2),$$

which satisfy the end conditions $\eta_1(t_1) = \eta_1(t_2) = f_\sigma(t_1) = f_\sigma(t_2) = 0$, and the set of differential equations $\Phi_\beta(\eta_1, \eta_1') = Q_{\beta x_i} \eta_i + Q_{\beta x_i'} \eta_i' = 0$ and $G_\sigma(\eta_1, \eta_1') - f_\sigma' = G_{\sigma x_i} \eta_i + G_{\sigma x_i'} \eta_i' - f_\sigma' = 0$. The function $2\omega(\eta, \eta')$ is the quadratic form $H_{x_1 x_j} \eta_1 \eta_j + 2H_{x_1 x_j'} \eta_1 \eta_j' + H_{x_1' x_j'} \eta_1' \eta_j'$. From the consideration of this accessory minimum problem we obtain the accessory differential equations

$$(5:1) \quad \frac{d}{dt} \Omega \eta_i - \Omega \eta_i = 0, \quad \Phi_\beta(\eta_i, \eta_i') = 0, \quad G_\sigma(\eta_i, \eta_i') - f_\sigma' = 0,$$

where $\Omega(\eta, \eta'; \epsilon, \nu) = \omega(\eta, \eta') + \epsilon_\beta \Phi_\beta(\eta, \eta') + \nu_\sigma G_\sigma(\eta, \eta')$.

By a special solution of the equations (5:1) we shall mean a solution for which the components $\eta_1(t)$, $f_\sigma(t)$ satisfy $x_1' \eta_1' + z_\sigma' f_\sigma' = \text{constant}$. If the arc E satisfies the condition I and along E the determinant

$$R = \begin{vmatrix} H_{x_1' x_j'} & 0 & Q_{\beta x_1'} & G_{\sigma x_1'} & x_1' \\ 0 & 0 & 0 & -I & G_\sigma \\ Q_{\beta x_1'} & 0 & 0 & 0 & 0 \\ G_{\sigma x_1'} & -I & 0 & 0 & 0 \\ x_1' & G_\sigma & 0 & 0 & 0 \end{vmatrix} \neq 0,$$

and the components $\eta_1(t), f_\sigma(t)$ of a special solution of (5:1) all vanish at two distinct values of t , say t_3 and t_4 , but are not identically zero between t_3 and t_4 , then the points $x_{31} = x_1(t_3)$ and $x_{41} = x_1(t_4)$ will be said to be conjugate points for the parametric isoperimetric problem of Lagrange. If the arc E has no point x_3 conjugate to x_1 between x_1 and x_2 or at x_2 then we shall say that E satisfies the condition IV'.

THEOREM 5:1. If the arc $E: x_1 = x_1(t), (t_1 \leq t \leq t_2)$ satisfies the conditions I, II', III', IV', then there exists a neighborhood of the arc E such that E minimizes the integral J in the class of all the curves of K which lie in this neighborhood and pass through the two fixed points x_{11} and x_{21} .

The neighborhood referred to in the theorem is one of restricted type as mentioned in the Introduction.

In order to give meaning to the statement of the theorem that the arc E satisfies the condition IV' we must show that the determinant $R \neq 0$ along E . Let us introduce the auxiliary dependent variables

$$z_\sigma = \int_{t_1}^t g_\sigma(x, x') dt.$$

Then the arc $\bar{E}: x_1 = x_1(s), z_\sigma = z_\sigma(s), (s_1 \leq s \leq s_2)$ is admissible in non-parametric space, has no corners, and satisfies the end conditions $s_1 = x_1(s_1) - x_{11} = x_1(s_2) - x_{21} = z_\sigma(s_1) = z_\sigma(s_2) - \ell_\sigma = 0$ and the set of differential equations $\mathcal{Q}_\beta(x, x') = 0, g_\sigma(x, x') - z'_\sigma = 0$, and $x_1' x_1' + z'_\sigma z'_\sigma = 1$. From the condition III' it follows that along $\bar{E}$ the quadratic form

$$r_1 H_{x_1' x_j'}(x, x'; \lambda(x), \mu) r_j > 0$$

for all $(r_1, \rho_\sigma) \neq (0, 0)$ satisfying $\mathcal{Q}_{\beta x_1'}(x, x') r_1 = 0$,

$g_{\sigma x_1'}(x, x') r_1 - \rho_\sigma = 0$, and $x_1' r_1 + z_\sigma' \rho_\sigma = 0$. It follows

that the determinant $R \neq 0$ along E .¹ It follows that the arc $\bar{E}$ satisfies the conditions I, II', III', IV' corresponding to the minimum problem in non-parametric space. The transversality condition in non-parametric space requires that the multiplier of the differential expression $x_1'x_1' + z_\sigma' z_\sigma' - 1$ be identically zero along $\bar{E}$.² It is clear, then, that the arc $\bar{E}$ satisfies the transversality condition. From a theorem due to Bliss³ it follows that $E_H(x, x', u; \lambda, \mu) > 0$ for all sets $(x, x', u; \lambda, \mu)$ for which (x, x') and (x, u) are admissible and satisfy $Q_\beta = 0$, $u \neq x'/\|x'\|$, and the points $(x, x'; \lambda, \mu)$ are in a suitably chosen neighborhood of such points on the arc E . It should be noted that the theorem of Bliss referred to above does not apply directly to the general problem considered here. But one can easily generalize it to apply to the case here studied by making use of Lemma 3:1 and the fact that the totality of directions q such that (x_0, q) are admissible and $Q_\beta(x_0, q) = 0$ constitute a closed set. From a known sufficiency theorem⁴ it follows that there exists a neighborhood of the arc $\bar{E}$ in non-parametric space such that the arc $\bar{E}$ minimizes the integral J in the class of all admissible curves in non-parametric space which satisfy the end conditions $s_1 = x_1(s_1) - x_{11} = x_1(s_2) - x_{21} = z_\sigma(s_1) = z_\sigma(s_2) - k_\sigma = 0$, the set of differential equations $Q_\beta(x, x') = 0$, $g_\sigma(x, x') - z_\sigma' = 0$, $x_1'x_1' + z_\sigma' z_\sigma' = 1$, and lie in this neighborhood of $\bar{E}$.

¹Bliss, American Journal of Mathematics, vol. 52 (1930), p. 735.

²Ibid., p. 698.

³Bliss, Transactions of the American Mathematical Society, vol. 15 (1914), p. 375.

⁴Morse, Transactions of the American Mathematical Society, vol. 37 (1935), p. 159.

By Theorems 4:2 and 4:1 we then conclude that there exists a neighborhood of the arc E such that the arc E minimizes the integral J in the class of all the curves of K which lie in this neighborhood and pass through the two fixed points x_{11} and x_{21} .

6. Osgood Theorems. In this section we shall develop two generalized Osgood theorems applicable to the parametric isoperimetric Lagrange problem. We shall make the same assumptions about the functions $f(x, x')$, $g_r(x, x')$ and $\mathcal{Q}_\beta(x, x')$ as in Section 2. It should be noted that as a corollary to Theorem 3:1 we can state the following lemma. The proof follows directly from that given for Theorem 3:1.

LEMMA 6:1. Let $\lambda_\beta(x)$ be a set of continuous functions of x for all x in A , and μ_r a set of constants. If $E_H(x, q, u; \lambda(x), \mu) \geq 0$ for all admissible (x, q) and $(\bar{x}, u)$ satisfying $\mathcal{Q}_\beta = 0$, then the integral J is lower semi-continuous at every curve in K with respect to every class of curves of K of uniformly bounded lengths.

THEOREM 6:1. Let C_0 be the unique minimizing curve of the integral J in a subclass K_0 of the class K . Let the class $\bar{K}_0$ of curves in enlarged space corresponding to the curves of K_0 be a closed class. Let there exist a set of multipliers $\lambda_\beta(x), \mu_\sigma$ and a vector function $r(x)$, where the $\lambda_\beta(x)$ and the vector function $r(x)$ are defined and continuous on C_0 and the μ_σ are constants, such that $E_H(x_0, r(x_0), q; \lambda(x_0), \mu) > 0$ for all (x_0, q) such that $\mathcal{Q}_\beta(x_0, q) = 0$ and $q \neq r(x_0)$. Let $E_H(x_0(s), x_0'(s), q; \lambda(x_0(s)), \mu) > 0$ for almost all s and all $(x_0(s), q)$ such that $\mathcal{Q}_\beta(x_0(s), q) = 0$ and $q \neq x_0'(s)/\|x_0'(s)\|$, s being the arc-length measured on C_0 from its initial point to a generic point on it. Further, let $E_H(x, q, u; \lambda(x), \mu) \geq 0$ for all admissible elements (x, q) and

$(\bar{x}, u)$ satisfying $\mathcal{Q}_\beta = 0$. Then there exists a $\rho > 0$ such that
for every $0 < \rho_1 < \rho$ there exists an $\eta > 0$ such that
 $J(C) - J(C_0) > \eta$ for all C in K_0 which lie in $(C_0)_\rho$ but not in
 $(C_0)_{\rho_1}$.

Let us introduce the auxiliary dependent variables

$$z_\sigma = \int_{t_1}^t g_\sigma(x, x') dt.$$

Then the curve $\bar{C}_0: x_1 = x_{01}(t), z_\sigma = z_{0\sigma}(t), (t_1 \leq t \leq t_2)$ is the unique minimizing curve in the closed class $\bar{K}_0$ of curves $\bar{C}: x_1 = x_1(t), z_\sigma = z_\sigma(t), (t_1 \leq t \leq t_2)$ which correspond to the curves of K_0 . By Theorem 3:2 for the parametric problem corresponding to a fixed $\delta > 0$ there exists an $\epsilon > 0$ and a ρ_2 neighborhood of $\bar{C}_0$ such that $J(\bar{C}) - J(\bar{C}_0) > \epsilon$ for all $\bar{C}$ in $\bar{K}_0$ and in $(\bar{C}_0)_{\rho_2}$ for which $L(\bar{C}) - L(\bar{C}_0) > \delta$. Since the length function is lower semi-continuous, there exists a neighborhood of $\bar{C}_0$ such that $|L(\bar{C}) - L(\bar{C}_0)| \leq \delta$ for every curve in this neighborhood for which $L(\bar{C}) - L(\bar{C}_0) \leq \delta$. Let $\bar{\rho}$ be the smaller of the two neighborhoods of $\bar{C}_0$ mentioned above. Let $0 < \bar{\rho}_1 < \bar{\rho}$, and let us define the class $\bar{K}_1$ as the subclass of $\bar{K}_0$ consisting of all the curves of $\bar{K}_0$ which lie in $(\bar{C}_0)_{\bar{\rho}}$ but not in $(\bar{C}_0)_{\bar{\rho}_1}$ for which $L(\bar{C}) - L(\bar{C}_0) \leq \delta$. This class is closed. Select a minimizing sequence for the integral J in $\bar{K}_1$. Since the lengths of this sequence are uniformly bounded, there exists a rectifiable curve of accumulation. Since $\bar{K}_1$ is closed, this curve belongs to $\bar{K}_1$. From the lower semi-continuity implied by Lemma 6:1, this curve $\bar{C}_1$ minimizes the integral J in the class $\bar{K}_1$. Now, $\bar{C}_1$ is distinct from $\bar{C}_0$. Hence $J(\bar{C}_1) > J(\bar{C}_0)$. Let $\mathcal{J}$ be the smaller of $|J(\bar{C}_1) - J(\bar{C}_0)|/2$ and ϵ . Then for every curve $\bar{C}$ in $\bar{K}_1$, $J(\bar{C}) - J(\bar{C}_0) \geq J(\bar{C}_1) - J(\bar{C}_0) > \mathcal{J} > 0$. By Lemma 4:2 there exists a ρ_3 and a $\mu > 0$ such that if C is in K_0 and in $(C_0)_{\rho_3}$, and $\bar{C}$ is

not in $(C_0)_{\bar{\rho}}$, then $L(C) - L(C_0) > \mu$. By Theorem 3:2 there exists a $\gamma > 0$ and a ρ_4 neighborhood of C_0 such that if C is in K_0 and in $(C_0)_{\rho_4}$ and $L(C) - L(C_0) > \mu$, then $J(C) - J(C_0) > \gamma$. Select ρ to be the smallest of $\bar{\rho}$, ρ_3 , and ρ_4 . Let $0 < \rho_1 < \rho$ and let C be a curve of K_0 lying in $(C_0)_\rho$ but not in $(C_0)_{\rho_1}$. There exists a $0 < \bar{\rho}_1 < \bar{\rho}$ such that C is not in $(\bar{C}_0)_{\bar{\rho}_1}$. Let $\eta > 0$ be the smaller of the J corresponding to $\bar{\rho}_1$ and γ . Then $J(C) - J(C_0) > \eta$, and the theorem is established.

The following theorem is the second of the generalized Osgood theorems.

THEOREM 6:2. Let C_0 be a curve of the class K which is the unique limit curve of every minimizing sequence of the class K . Let $J(C)$ be lower semi-continuous at C_0 with respect to the class K . Then for every neighborhood of C_0 , there exists an $\eta > 0$ such that $J(C) - J(C_0) > \eta$ for every C in K not in this neighborhood of C_0 .

Select a neighborhood of C_0 , and let K_1 denote the class of curves belonging to K which do not lie in this neighborhood of C_0 . Select a minimizing sequence for the integral J in the class K_1 . Since every such minimizing sequence could not be one for J in the class K , the greatest lower bound for J in K_1 must necessarily be greater than the greatest lower bound for J in K . Define $\eta > 0$ to be equal to one-half the difference between the two greatest lower bounds. Then for every C in K_1 , $J(C) - J(C_0) > \eta$, and the theorem is established.

BIBLIOGRAPHY

1. Tonelli, Fondamenti di calcul delle variazioni, I (Bologna, 1921), II (1923).
2. McShane, Dissertation, Chicago (1930).
3. Lindeberg, Mathematische Annalen, LXVII (1909), pp. 340-354.
4. Fréchet, Transactions of the American Mathematical Society, vol. 6 (1905), pp. 435-449.

